The kinetic equation
(Lecture 11)

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In the preceding lectures we focused our attention on a single particle motion. In this lecture, we will introduce formalism for treating an ensemble of particles circulating in an accelerator ring.
We start from considering a simple case of one degree of freedom with the canonically conjugate variables $q$ and $p$.

A large ensemble of particles (think about a particle beam) with each particle having various values of $q$ and $p$ constitutes a “cloud” in the phase space.
Distribution function in phase space and kinetic equation

Let us consider an infinitesimally small region in phase space $dq \times dp$ and let the number of particles of the beam at time $t$ in this. Mathematically infinitesimal phase element should be physically large enough to include many particles, $dN \gg 1$. We define the distribution function of the beam $f(q, p, t)$ such that

$$dN(t) = f(q, p, t)dp \, dq. \quad (11.1)$$

We can say that the distribution function gives the density of particles in the phase space.
Particles travel from one place in the phase space to another, and the distribution function evolves with time. Our goal is to derive a *kinetic equation* that governs this evolution. In this derivation, we will assume that particles' motion is Hamiltonian. Consider an infinitesimally small region of the phase space.
The number of particles in this region at time $t$ is given by Eq. (11.1). At time $t + dt$ this number will change because of the flow of particles through the boundaries. Due to the flow in the $q$-direction the number of particles that flow in through the left boundary is

$$f(q, p, t) \times dp \dot{q}(q, p, t) \times dt$$  \hspace{1cm} (11.2)$$

and the number of particles that flow out through the right boundary is

$$f(q + dq, p, t) \times dp \dot{q}(q + dq, p, t) \times dt.$$  \hspace{1cm} (11.3)$$
Distribution function in phase space and kinetic equation

Similarly, the number of particles which flow in through the lower horizontal boundary is

$$ f(q, p, t) \times dq \dot{p}(q, p, t) \times dt $$

(11.4)

and the number of particles that flow out through the upper horizontal boundary is

$$ f(q, p + dp, t) \times dq \dot{p}(q, p + dp, t) \times dt. $$

(11.5)

The number of particles in the volume $dq \times dp$ is now changed

$$ dN(t + dt) - dN(t) $$

$$ = [f(q, p, t + dt) - f(q, p, t)] dp \ dq $$

$$= f(q, p, t) dp \dot{q}(q, p, t) dt - f(q + dq, p, t) dp \dot{q}(q + dq, p, t) dt $$

$$+ f(q, p, t) dq \dot{p}(q, p, t) dt - f(q, p + dp, t) dq \dot{p}(q, p + dp, t) dt. $$

(11.6)
Distribution function in phase space and kinetic equation

Dividing this equation by \( dp \, dq \, dt \) and expanding in Taylor’s series (keeping only linear terms in \( dp, dq, dt \)) gives the following equation

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial q} \left[ \dot{q}(q, p, t) f \right] + \frac{\partial}{\partial p} \left[ \dot{p}(q, p, t) f \right] = 0 .
\]  

(11.7)

What we derived is the continuity equation for the function \( f \).
Incompressible Hamiltonian flow

Due to the Hamiltonian nature of the flow in the phase space a medium represented by a distribution function $f$ in *incompressible*. This follows from the Liouville theorem. Indeed, according to this theorem the volume of a space phase element does not change in Hamiltonian motion. Since the value of $f$ is the number of particles in this volume, and this number is conserved, $f$ within a *moving* elementary volume is also conserved. The density at a given point of the phase space $q, p$ however changes because other liquid elements arrive at this point at a later time.
Mathematically, the fact of incompressibility is reflected in the following transformation of the continuity equation (11.7). Let us take into account the Hamiltonian equations for $\dot{q}$ and $\dot{p}$:

$$
\frac{\partial}{\partial q} \dot{q}(q, p, t) = \frac{\partial}{\partial q} \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \frac{\partial H}{\partial q} = -\frac{\partial}{\partial p} \dot{p}(q, p, t), \quad (11.8)
$$

which allows to rewrite Eq. (11.7) as follows

$$
-\frac{\partial f}{\partial t} + \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} = 0. \quad (11.9)
$$
Distribution function in phase space and kinetic equation

In accelerator physics this equation is often called the \textit{Vlasov} equation. It is a partial differential equation which is not easy to solve in most of the cases. It is however extremely useful for studying many effects in accelerators that involve interaction between the particles of the beam. Note, that using the formalism of Poisson brackets, we can also write the Vlasov equation as

$$\frac{\partial f}{\partial t} + \{H, f\} = 0. \quad (11.10)$$

See (3.31). This means that

$$\frac{df}{dt} = 0$$
Distribution function in phase space and kinetic equation

In case of $n$ degrees of freedom, with the canonical variables $q_i$ and $p_i$, $n = 1, 2, \ldots, n$, the distribution function $f$ is defined as a density in $2n$-dimensional phase space and depends on all these variables, $f(q_1, \ldots, p_1, \ldots, t)$. The Vlasov equation takes the form

$$-\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left( \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} \right) = 0.$$  \hspace{1cm} (11.11)

Sometimes it is more convenient to normalize $f$ by $N$, then the integral of $f$ over the phase space is equal to one.
Integration of the kinetic equation along trajectories

We have stated above that the distribution function is constant within a moving infinitesimal element of phase space “liquid”. We will now prove it.
Consider a trajectory in the phase space, and calculate the difference of $f$ at two close points on this trajectory.
Integration of the kinetic equation along trajectories

We have

\[ df = f(q + dq, p + dp, t + dt) - f(q, p, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp. \] (11.12)

Remember that the two points are on the same trajectory, hence, \( dq = \dot{q} dt = \frac{\partial H}{\partial p} dt \) and \( dp = \dot{p} dt = -\frac{\partial H}{\partial q} dt \). We find

\[ df = \frac{\partial f}{\partial t} dt - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} dt + \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} dt = 0. \] (11.13)

On the last step we invoked Eq. (11.9). We proved that the function \( f \) is constant along the trajectories.
Integration of the kinetic equation along trajectories

The above statement opens up a way to find solutions of the Vlasov equation if the phase space orbits are known. Let $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ be solutions of the Hamiltonian equations of motion with initial values $q_0$ and $p_0$ at $t = 0$, and $F(q_0, p_0)$ be the initial distribution function at $t = 0$. Then the solution of the Vlasov equation is given by the following equations

$$f(q, p, t) = F(q_0(q, p, t), p_0(q, p, t)), \quad (11.14)$$

where the functions $q_0(q, p, t)$ and $p_0(q, p, t)$ are obtained as inverse functions from equations

$$q = q(q_0, p_0, t), \quad p = p(q_0, p_0, t). \quad (11.15)$$
Steady state solutions of the kinetic equation

One of the powerful methods of solving the Vlasov equation is based on a judicious choice of canonical variables. Let us use canonical variables $J$ and $\phi$. Then our kinetic equation is

$$\frac{\partial f}{\partial s} + \frac{\partial \hat{H}}{\partial J} \frac{\partial f}{\partial \phi} - \frac{\partial \hat{H}}{\partial \phi} \frac{\partial f}{\partial J} =$$

$$\frac{\partial f}{\partial s} + \frac{\partial \hat{H}}{\partial J} \frac{\partial f}{\partial \phi} = 0. \quad (11.16)$$

We see from this equation that any function $f$ that depends only on $J$ satisfies the equation $\partial f/\partial s = 0$, and hence is a steady state solution.
Steady state solutions of the kinetic equation

The particular dependence $f(J)$ is determined by various other processes in the ring. In many cases, a negative exponential dependence $f$ versus $J$ is a good approximation

$$f = \text{const } e^{-J/\varepsilon_0} = \text{const } \exp \left(-\frac{1}{2\beta \varepsilon_0} \left[ x^2 + (\beta x' + \alpha x)^2 \right] \right).$$

(11.17)

The quantity $\varepsilon_0$ is called the beam emittance. It is an important characteristic of the beam quality.
Phase mixing and decoherence

Consider an ensemble of linear oscillators with the frequency $\omega$, whose motion is described by the Hamiltonian

$$H(x, p) = \frac{p^2}{2} + \omega^2 \frac{x^2}{2}.$$ \hspace{1cm} (11.18)

The distribution function $f(x, p, t)$ for these oscillators satisfy the Vlasov equation

$$\frac{\partial f}{\partial t} - \omega^2 x \frac{\partial f}{\partial p} + p \frac{\partial f}{\partial x} = 0.$$ \hspace{1cm} (11.19)

We can easily solve this equation. The trajectory of an oscillator with the initial coordinate $x_0$ and momentum $p_0$ is

$$x = x_0 \cos \omega t + \frac{p_0}{\omega} \sin \omega t$$

$$p = -\omega x_0 \sin \omega t + p_0 \cos \omega t.$$ \hspace{1cm} (11.20)
Phase mixing and decoherence

Inverting these equations, we find

\[ x_0 = x \cos \omega t - \frac{p}{\omega} \sin \omega t \]
\[ p_0 = \omega x \sin \omega t + p \cos \omega t . \quad (11.21) \]

If \( F(x, p) \) is the initial distribution function at \( t = 0 \), then, according to Eq. (11.14) we have

\[ f(x, p, t) = F \left( x \cos \omega t - \frac{p}{\omega} \sin \omega t, \omega x \sin \omega t + p \cos \omega t \right) . \quad (11.22) \]

This solution describes rotation of the initial distribution function in the phase space. An initially offset distribution function results in collective oscillations of the ensemble with the betatron period. A mismatched distribution oscillates at half the betatron period.
Phase mixing and decoherence

A more interesting situation occurs if there is a frequency spread in the ensemble. Let us assume that each oscillator is characterized by some parameter $\delta$ (that does not change with time), and $\omega$ is a function of $\delta$, $\omega(\delta)$.

$$H(x, p, \delta) = \frac{p^2}{2} + \omega(\delta)^2 \frac{x^2}{2}. \quad (11.23)$$

We then have to add $\delta$ to the list of the arguments of $f$ and $F$, and Eq. (11.22) becomes

$$f(x, p, t, \delta) = F\left( x \cos \omega(\delta)t - \frac{p}{\omega(\delta)} \sin \omega(\delta)t, \omega(\delta)x \sin \omega(\delta)t + p \cos \omega(\delta)t, \delta \right). \quad (11.24)$$
Phase mixing and decoherence

To find the distribution of oscillators over $x$ and $p$ only one has to integrate $f$ over $\delta$

$$\hat{f}(x, p, t) = \int_{-\infty}^{\infty} d\delta \, f(x, p, t, \delta). \quad (11.25)$$

The behavior of the integrated function $\hat{f}$ is different from the case of constant $\omega$ at large times, even if the spread in frequencies $\Delta \omega$ is small. For $t \gtrsim 1/\Delta \omega$ the oscillators smear out over the phase. This effect is called the *phase mixing* and it results in *decoherence* of collective oscillations of the ensemble of oscillators.
Phase mixing and decoherence

In the limit $t \to \infty$ an initial distribution approaches a steady state which does not depend on time. We can find it using the action-angle variables.

$$H(J, \delta) = \omega(\delta)J \quad (11.26)$$

with

$$J = \frac{1}{2\omega} \left( p^2 + \omega(\delta)^2 x^2 \right). \quad (11.27)$$

The Vlasov equation in $J - \phi$ coordinates is

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial J} \frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial t} + \omega(\delta) \frac{\partial f}{\partial \phi} = 0. \quad (11.28)$$

In steady state $\partial f/\partial t = 0$, hence $f$ only depends on $J$. This distribution should be $f_{eq}(J) = (1/2\pi) \int d\phi \hat{f}(\phi, J, t = 0)$, which is an orbit integral.