Synchrotron radiation  
(Lecture 20)

February 3, 2016
Lecture outline

We will consider a relativistic point charge ($\gamma \gg 1$) moving in a circular orbit of radius $\rho$. Our goal is to calculate the synchrotron radiation of this charge. Using the Liénard-Wiechert potentials we first find the fields at a large distance from the charge in the plane of the orbit. We then discuss properties of the synchrotron radiation using a more general result for the angular dependence of the spectral intensity of the radiation.
Synchrotron radiation pulses in the plane of the orbit

An observer is located in point O in the plane of the orbit in the far zone. The observer will see a periodic sequence of pulses of electromagnetic radiation with the period equal to the revolution period of the particle around the ring, \( \omega_r = \beta c / \rho \) is the revolution frequency. Each pulse is emitted from the region \( x \approx z \approx 0 \).
Synchrotron radiation pulses in the plane of the orbit

Main steps in the derivation

- Use the plane wave approximation for the radiation field (and replace $n$ with $\hat{z}$):

$$B = -\frac{1}{c} \hat{z} \times \frac{\partial A}{\partial t}, \quad E = -c \hat{z} \times B$$

- Denote the retarded time by $\tau$, so that $R(\tau) = c(t - \tau)$, use $R(\tau) \approx r - \rho \sin \omega_r \tau$. At time $\tau = 0$ the particle is located at $x = z = 0$, $\rho \sin \omega_r \tau$ is approximately equal to the $z$ coordinate at time $\tau$, hence $R \approx r - z$.

- In the expression for $A$, further approximate the factor $R$ in the denominator by $R \approx r$, yielding

$$A(r, t) = \frac{Z_0 q}{4\pi r} \frac{\beta(t_{\text{ret}})}{1 - \beta(t_{\text{ret}}) \cdot n}$$

Quantities depending on $t_{\text{ret}}$ are not approximated in this way.
Field in the synchrotron pulse

The final result

\[ B_y = \frac{Z_0 q}{\pi r \rho} \frac{\gamma^{-2} - \xi^2}{(\xi^2 + \gamma^{-2})^3}, \quad t = \frac{r}{c} + \frac{\rho}{c} \left( \frac{1}{2\gamma^2} \xi + \frac{1}{6} \xi^3 \right) . \] (20.1)

The variable \( \xi = c \tau \rho = \omega_r \tau / \beta \) has a simple physical meaning—it is roughly the angle on the orbit from which radiation that arrives at the observation point \( O \) originates [remember our original concept of waves emitted by an accelerated particle]. Polarization: the electric field of radiation is in the plane of the orbit.

Introduce the dimensionless time variable \( \hat{t} = (\gamma^3 c / \rho)(t - r / c) \) and the dimensionless magnetic field \( \hat{B} = (\pi r \rho / Z_0 q \gamma^4)B_y \):

\[ \hat{B} = \frac{1 - \zeta^2}{(\zeta^2 + 1)^3}, \quad \hat{t} = \frac{1}{2} \zeta + \frac{1}{6} \zeta^3, \] (20.2)

where \( \zeta = \xi \gamma \).
The characteristic width of the pulse $\Delta \hat{t} \sim 1$, which means that the duration of the pulse in physical units

$$\Delta t \sim \frac{\rho}{c\gamma^3}.$$  \hfill (20.3)

The spectrum of frequencies presented in the radiation is $\Delta \omega \sim c\gamma^3/\rho$. 
Fourier transformation of the radiation field and the radiated power

The power $P$ radiated in unit solid angle $d\Omega$ in the $x$-$z$ plane is, using the plane wave approximation,

$$\frac{dP}{d\Omega} = r^2 S \cdot n = \frac{r^2 c}{\mu_0} B_y^2(t) \quad (20.4)$$

The total energy flux $\mathcal{W}$ in this plane is

$$\frac{d\mathcal{W}}{d\Omega} = r^2 \int_{-\infty}^{\infty} dt \, S(t).$$

We consider the spectrum of the radiation. Use Parseval's theorem:

$$\int_{-\infty}^{\infty} dt \, B_y(t)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega |\tilde{B}_y(\omega)|^2 = \frac{1}{\pi} \int_{0}^{\infty} d\omega |\tilde{B}_y(\omega)|^2,$$

where

$$\tilde{B}_y(\omega) = \int_{-\infty}^{\infty} dt \, B_y(t) e^{i\omega t}.$$
Fourier transformation of the radiation field

We introduce the energy radiated per unit frequency interval per unit solid angle as

$$\frac{d^2 \mathcal{W}}{d\omega d\Omega} = \frac{r^2 c^2}{\pi Z_0} |\tilde{B}_y(\omega)|^2,$$  \hspace{1cm} (20.5)

so that the total energy radiated per unit solid angle is

$$\frac{d\mathcal{W}}{d\Omega} = \int_0^\infty d\omega \frac{d^2 \mathcal{W}}{d\omega d\Omega}.$$  \hspace{1cm} (20.6)

$$[\frac{d\mathcal{W}}{d\Omega} \text{ and } \frac{d^2 \mathcal{W}}{d\omega d\Omega} \text{ are not derivatives of the function } \mathcal{W}, \text{ just a notation.}]$$
Fourier transformation of the radiation field

The function $\tilde{B}_y(\omega)$ is calculated in the Lecture notes. The spectrum is

$$\frac{d^2 \mathcal{W}}{d\omega d\Omega} = \frac{q^2 Z_0}{12\pi^3} \left( \frac{\rho \omega}{c} \right)^2 \left( \frac{1}{\gamma^2} \right)^2 K_{2/3}^2 \left( \frac{\omega}{2\omega_c} \right),$$

(20.7)

where $K_{2/3}$ is the MacDonald function, and the critical frequency

$$\omega_c = \frac{3c\gamma^3}{2\rho}.$$  

(20.8)

The dominant part of the spectrum is in the region $\omega \sim \omega_c$. For NSLS-II $\omega \sim \omega_c$ corresponds to the wavelength of 0.5 nm or 2.4 keV photon energy.
Synchrotron radiation for $\psi \neq 0$

In a more general case of radiation at an angle $\psi \neq 0$ the calculation is more involved. We will summarize some of the results of this general case.
Synchrotron radiation for $\psi \neq 0$

A more general formula valid for $\psi \neq 0$ is

$$\frac{d^2 \mathcal{W}}{d\omega d\Omega} = \frac{q^2 Z_0}{12\pi^3} \left( \frac{\rho \omega}{c} \right)^2 \left( \frac{1}{\gamma^2} + \psi^2 \right)^2 \left[ K_{2/3}(\chi) + \frac{\psi^2}{1/\gamma^2 + \psi^2} K_{1/3}(\chi) \right],$$

(20.9)

where

$$\chi = \frac{\omega \rho}{3c} \left( \frac{1}{\gamma^2} + \psi^2 \right)^{3/2} = \frac{\omega}{2\omega_c} \left( 1 + \psi^2 \gamma^2 \right)^{3/2}.$$

(20.10)

This result was obtained by J. Schwinger in 1949. Setting $\psi = 0$ we recover Eq. (20.7).
Synchrotron radiation for $\psi \neq 0$

The Bessel functions falls off when the argument $\chi \gg 1$.

The spectrum is strongly correlated with the angle. What is the angular spread in $\psi$? The dominant part of the radiation is in the region $\chi \lesssim 1$. If $\omega \sim \omega_c$, then $\psi \lesssim 1/\gamma$. For lower frequencies, the angle is larger:

$$\psi \sim \frac{1}{\gamma} \left(\frac{\omega_c}{\omega}\right)^{1/3} \sim \left(\frac{\lambda}{\rho}\right)^{1/3} \quad (20.11)$$
Synchrotron radiation for $\psi \neq 0$

The two terms in the square brackets correspond to different polarizations of the radiation. The first one is the so called $\sigma$-mode, it has polarization with $E_x$ and $B_y$. The second one has the polarization with the electric field $E_y$ and the magnetic field $B_x$; it is called the $\pi$ mode.
Synchrotron radiation for $\psi \neq 0$

The radiation is localized at small angles $\psi$.

when integrating over $d\Omega$, in addition to integration over the angle $\psi$, one should include integration over the angle $\theta$,

\[
\frac{d\mathcal{W}}{d\omega} = \int d\Omega \frac{d^2\mathcal{W}}{d\omega d\Omega} = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \cos \psi d\psi \frac{d^2\mathcal{W}}{d\omega d\Omega}
\]

\[
\approx 2\pi \int_{-\infty}^{\infty} d\psi \frac{d^2\mathcal{W}}{d\omega d\Omega}
\]
Synchrotron radiation for $\psi \neq 0$

Integration over the angle gives the frequency distribution

$$\frac{d\mathcal{W}}{d\omega} = \frac{2\pi \rho}{c} \frac{q^2 \gamma Z_0 c}{9\pi \rho} S \left( \frac{\omega}{\omega_c} \right),$$  \hspace{1cm} (20.12)

where

$$S(x) = \frac{27x^2}{16\pi^2} \int_{-\infty}^{\infty} d\tau \left(1 + \tau^2\right)^2$$

$$\cdot \left[ K_{2/3}^2 \left( \frac{x}{2} (1 + \tau^2)^{3/2} \right) + \frac{\tau^2}{1 + \tau^2} K_{1/3}^2 \left( \frac{x}{2} (1 + \tau^2)^{3/2} \right) \right]$$

$$= \frac{9\sqrt{3}}{8\pi} \int_{x}^{\infty} K_{5/3}(y) \, dy.$$

The last expression is not easy to derive, but it is the most common definition used.
Synchrotron radiation for $\psi \neq 0$

Plot of function $S$.

The function $S$ is normalized to one: $\int_0^\infty d\omega S(\omega) = 1$. 
Synchrotron radiation for $\psi \neq 0$

For small and large values of the argument we have the asymptotic expressions

$$S(x) = \frac{27}{8\pi} \frac{\sqrt{3}}{2^{1/3}} \Gamma \left( \frac{5}{3} \right) x^{1/3}, \quad x \ll 1$$

$$S = \frac{9}{8} \sqrt{\frac{3}{2\pi}} \sqrt{x} e^{-x}, \quad x \gg 1. \quad (20.13)$$
Total radiated power

Integrating $d\mathcal{W}/d\omega$ over all frequencies, we will find the total energy $W_r$ radiated in one revolution

$$W_r = \int_{0}^{\infty} d\omega \frac{d\mathcal{W}}{d\omega} = \frac{2\pi \rho}{c} \cdot \frac{q^2 \gamma Z_0 c}{9\pi \rho} \omega_c.$$  \hspace{1cm} (20.14)

The radiation power (energy radiation per unit time) by a single electron is

$$\mathcal{P} = \frac{W_r}{2\pi \rho/c} = \frac{Z_0 c q^2 \gamma}{9\pi \rho} \omega_c = \frac{2r_0 mc^2 \gamma^4 c}{3\rho^2}.$$  \hspace{1cm} (20.15)

Note that $\mathcal{P}/c$ is the energy radiated by one electron per unit length of path.

Number of photons per unit bandwidth per unit time is:

$$\frac{d^2 N_{ph}}{dt \, d\omega} = \frac{1}{\hbar \omega} \frac{d\mathcal{W}}{d\omega} \frac{1}{2\pi \rho/c} = \frac{8}{27} \alpha \frac{1}{\gamma^2} \frac{\omega_c}{\omega} S \left( \frac{\omega}{\omega_c} \right).$$  \hspace{1cm} (20.16)
Let’s calculate the power of synchrotron radiation from dipole magnets in NSLS-II. The energy 3 GeV, bending magnetic field 0.4 T, current $I = 400$ mA. Bending radius $\rho = p/eB = 25$ m, $C = 780$ m.

The critical frequency $\omega_c = 3.6 \times 10^{18}$ 1/s corresponding to the wavelength $\lambda = 0.5$ nm and the photon energy 2.4 keV.

Radiated power by one electron $8.8 \times 10^{-8}$ W [note that an electron radiates this power only inside the dipole and dipoles occupy a small fraction of the ring]. Number of electrons in the ring $N = (C/c)I/e = 8.1 \times 10^{12}$ which the gives total power of 0.14 MW. [There is additional radiation due to the wigglers in the ring].